ARL 65-71
APRIL 1965

Aerospace Research Laboratories

ON PRODUCTS AND QUOTIENTS OF RANDOM VARIABLES

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OFFICE OF AEROSPACE RESEARCH

United States Air Force

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500 - May 1965 - 448-41-894

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Contract AF 33(615)-1023

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UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

FOREWORD

This report was prepared by Dr. Robert S. Dezur and James D. Donahue, Electronics and Mathematics Laboratory, The Martin Company, Denver, Colorado, under Contract AF 33(615)-1023 for the Aerospace Research Laboratories (ARL), Office of Aerospace Research, United States Air Force. The research reported herein was accomplished under Task 7071-01, Research in Mathematical Statistics and Probability Theory, of Project 7071, Mathematical Techniques of Aerodynamics. Dr. P. R. Krishnaiah of the ARL was the contract monitor.

ABSTRACT

This report, prepared in two parts, deals with products and quotients of random variables. In Part I, the distributions of quotients of indep .dent random variables are considered. In Part II, the distribution of the product of two (not necessarily independent) normally distributed random variates is investigated. The tables of this distribution are given in the Appendix.

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I. ON THE QUOTIENT OF RANDOM VARIABLES

1. Introduction

In a study of distributions of products and quotients of random variables it is sometimes necessary to determine possible component distributions when the composite distribution is known. Formally, at least, this involves a study of linear integral equations of the first kind. For the quotient, in particular, suppose x_1 and x_2 are independent random variables with $f(x_1)$ and $g(x_2)$ their respective density functions. Setting $y_1 = x_1/x_2$ and $y_2 = x_2$, the density function for the quotient has the form $\varphi(y_1) = \int_{-\infty}^{\infty} f(y_1 y_2)\, g(y_2) \left| y_2 \right| dy_2$. A derivation of this formula as well as a general discussion of results in this area is given in [20].

A number of authors [22, 23, 31, 32, 33, 34, and others] have studied this problem in the case where the variates are assumed to be identically distributed. Their techniques, which can be called more or less "classical", involved the use of various transform theories — Mellin, Fourier, and others — but little was done to develop a general theory for the above equation. It was the original intent of this paper

Manuscript released by the authors December 1964 for publication as an ARL technical report.

to view this equation in operator form, i.e., $\varphi = F g$ (assuming f given), where F is a compact operator on an L^p space and attempt to develop a theory for linear integral equations of the first kind from the linear operator point of view that would be applicable to probability density functions. We have not been too successful as far as fruitful results are concerned. Some of the difficulties are discussed in Sec. 3 along with a possible application of a theory that is presently being developed for pairs of operators acting between Banach spaces.

In Sec. 2 we generalize a few known results assuming identical distributions as mentioned above (still from the classical point of view) and discuss an existence theorem. Essentially, the procedures and techniques of Laha have been followed here. In Appendix A two theorems resulting from a side investigation are presented - one is apparently not new but the proof seems particularly simple.

2. The Quotient of Independent Random Variables

We give here a few results and generalizations of known results involving the quotient of independent random variables. In parts of Sec. 2.1 and Sec. 2.2 "identically distributed" is also assumed.

2.1. A Necessary Condition for a Solution to Exist

Suppose x_1 and x_2 are identically distributed random variables over the real line with density function f. Under the transformation $y_1 = x_1/x_2$ and $y_2 = x_2$, see [20], the density function for the quotient x_1/x_2 is given by

$$\text{(A)} \qquad \varphi(y_1) = \int_{-\infty}^{\infty} f(y_1 y_2)\, f(y_2)\, |y_2|\, dy_2.$$

Note then that $\varphi\left(\frac{1}{y_1}\right) = \int_{-\infty}^{\infty} f(y_2/y_1)\, f(y_2)\, |y_2|\, dy_2$. For $y_1 \neq 0$, letting $y_2/y_1 = w$ so that $dy_2 = y_1 dw$ this becomes

$$\varphi\left(\frac{1}{y_1}\right) = \int_{-\infty}^{\infty} f(w)\, f(y_1 w)\, |y_1|\, |w|\, |y_1|\, dw$$

$$= |y_1|^2 \int_{-\infty}^{\infty} f(w)\, f(y_1 w)\, |w|\, dw$$

$$= |y_1|^2\, \varphi(y_1).$$

Note also that $\psi(0) = f(0)\, E\left[\,|y_2|\,\right]$. Viewing ψ as given in (A) above we have proved the following.

Theorem 1: Under the conditions above, if (A) has a solution then for $y_1 \neq 0$ it is necessary that $\psi\left(\frac{1}{y_1}\right) = y_1^{\,2}\psi(y_1)$; if not, no solution f exists. Moreover, the "search" for possible solutions can be narrowed down to those density functions f(x) such that $\psi(0) = f(0)\, E\left[\,|x|\,\right]$.

It is interesting to note in this result that $y_1^{\,2}\psi(y_1)$ is actually the density function for the random variable $1/y_1$. Note further that (A) cannot be solved if it is assumed that $\psi(y_1)$ is the normal density function. Referring to (A) again, suppose a symmetric solution is desired, i.e., $f(z) = f(-z)$. The equation then has the form

$$(B) \quad \psi(y_1) = 2\int_0^{\infty} f(y_1 y_2)\, f(y_2)\, y_2\, dy_2.$$

It is clear, first of all, that f symmetric implies ψ must also be symmetric. Also if ψ is given symmetric then

$$\int_{-\infty}^{\infty} |y_2|\, f(-y_1 y_2)\, f(y_2)\, dy_2 = \int_{-\infty}^{\infty} |y_2|\, f(y_1 y_2)\, f(y_2)\, dy_2$$

which implies that $f(-y_1 y_2) = f(y_1 y_2)$ a.e. Hence, in the a.e. sense at least, we have the following for (A) above.

Lemma 2: φ is symmetric if an only if f is symmetric.

Changing the form of (B) above,

$$\varphi(y_1) = 2\int_0^\infty y_2^{1/2} f(y_2)\, y_2^{1/2} f(y_1 y_2)\, dy_2 .$$

Letting $y_2^{1/2} f(y_2) = g(y_2)$ this becomes

$$2\int_0^\infty g(y_2)\, y_2^{1/2} f(y_1 y_2)\, dy_2$$

so that for $y_1 \geq 0$, $y_1^{1/2}\varphi(y_1) = 2\int_0^\infty g(y_2)\, y_1^{1/2}\, y_2^{1/2} f(y_1 y_2)\, dy_2$, i.e.,

$$\theta(y_1) = y_1^{1/2}\varphi(y_1) = 2\int_0^\infty g(y_2)\, g(y_1 y_2)\, dy_2 .$$

This is a form studied in [21]. Solving the above equation for g will also furnish a symmetric solution to the original equation.

It should be mentioned that Fox [21] carries out an analysis on the above form for $L'o,\infty)$ functions using Mellin and Fourier transform theory.

2.2. Some Generalizations Using Both Distribution and Density Functions

Laha [22] considers, in particular, an integral equation of the form (A) above, where y_1, the quotient of two independent, identically distributed random variables, is assumed to follow the Cauchy law. The general technique is to use distribution functions and Fourier transforms, the distribution functions assumed to be everywhere continuous to the right. This is a more general approach since the distribution function always exists.

The distribution function F for the random variable y is said to be symmetric (about 0) in case $F(y) = 1 - F(-y-0)$.

Lemma 1: Given the random variable x with distribution function $F(x)$ symmetric (about 0), the distribution function G of $|x|$ is given by

$$G(|x|) = \begin{cases} 2F(x) - 1, & x \geq 0 \\ 0, & \text{elsewhere}. \end{cases}$$

Proof: for $a \geq 0$,

$$G(a) = \Pr\left[\,|x| \leq a\,\right] = \Pr\left[-a \leq x \leq a\right] = F(a) - F(-a-0) = 2F(a)-1$$

since F is symmetric and the result follows.

The importance of the lemma is that the distribution function of x can be determined knowing only that of $|x|$ and we shall be able to relate this to the distribution of $\ln|x|$. Before doing this however we prove the following.

Lemma 2: Let u be a random variable following the Cauchy law, i.e., $f(u) = \frac{1}{\pi(1+u^2)}$. Then $z = \cot^{-1}u$ has a uniform distribution.

Proof: $\Pr\left[z \le a\right] = \Pr\left[\cot^{-1}u \le a\right] = 1 - \Pr\left[u \le \cot a\right]$

$$= 1 - \frac{1}{\pi}\int_{-\infty}^{\cot a} \frac{1}{1+u^2}\,du = 1 - \frac{1}{\pi}\left[(\pi/2 - a) + \pi/2\right] = \frac{a}{\pi}$$

so that the density function for z is equal to $\frac{1}{\pi}$, $0 \le z \le \pi$ and zero elsewhere. Using the above lemma, since z has a uniform distribution, and since we know its closed form characteristic function, see [25] for example, and moreover since the characteristic function of a function of a random variable, $g(w)$ say, is the mean value of $e^{i\,t\,g(w)}$ we can evaluate the following integral,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\,t\cot^{-1}w} \cdot \frac{1}{1+w^2}\,dw = e^{i\,t\pi/2}\;\frac{\sin t\pi/2}{t\pi/2}.$$

Theorem 3: Let x and y be independent, identically distributed random variables. Let $z = x/y$ and $G(z)$, the distribution function of z, be symmetric about 0. Suppose further that the square root of the characteristic function of $\ln|z|$ is absolutely integrable. Then $F(x)$, the distribution function of x (and y) is absolutely continuous and has a continuous density function $f(x) = F'(x) > 0$.

Proof: By a result of Laha [22], $F(x)$ is symmetric about the origin. Consequently the distribution function for $|x|$ is $G(x) = \begin{cases} 2F(x)-1, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$ by the above lemma. Let $E\left[e^{it\ln|z|}\right] = \varphi_z(t)$. Then since $\ln|z| = \ln|x| - \ln|y|$ we have

$$\varphi_x(t) \cdot \varphi_x(-t) = \varphi_z(t)$$

and hence
$$|\varphi_x(t)| = |\varphi_z(t)|^{1/2}.$$

By assumption $\int_{-\infty}^{\infty} |\varphi_z(t)|^{1/2}\, dt < \infty$ so that the characteristic function of $\ln|z|$, $\varphi_x(t)$, is absolutely integrable. By a theorem of Loeve [24] the distribution function of $\ln|x|$ is absolutely continuous and has a continuous density function. But since $F(\ln|x|) = G(|x|)$ it follows that $|x|$ has an absolutely continuous distribution function and a continuous density function. From above, then, so does x.

It is known that if z follows the Cauchy law, the characteristic function for $\ln|z|$ is $\operatorname{sech}(\pi t/2)$, a function which has a finite integral over the real line so that the above result holds for this particular distribution.

Theorem 4: To the assumptions in theorem 3 above add that arcot z has a uniform distribution. Then f, the density function for x, satisfies the integral equation

$$\int_0^{\infty} f(y)\, f(wy) y\, dy = \frac{k}{1+w^2}, \text{ where k is a constant.}$$

Proof: Our assumptions imply that f is also symmetric about zero and hence we recognize the above integral as $\frac{1}{2}\, g(w)$, where g is the density function for $w = x/y$. Let $u = \text{arcot } w$. The density function for u, h(u) say, is

$$h(u) = \begin{cases} \frac{1}{\pi}, & 0 \leq u \leq \pi \\ 0, & \text{elsewhere,} \end{cases}$$

so that for the distribution function H(u) we have

$$H(u) = \Pr\left[u \leq a\right] = 1 - \Pr\left[w \leq \cot a\right] = 1 - \int_{-\infty}^{\cot a} g(w)\, d(w) = \frac{a}{\pi},\ 0 \leq a \leq \pi.$$

Hence

$$\int_{-\infty}^{\cot a} \frac{1}{2}\, g(w)\, dw = \frac{1}{2} \cdot \frac{1}{\pi} (\pi - a)\,.$$

But

$$\int_{-\infty}^{\cot a} \frac{1}{1+w^2}\, dw = (\pi - a) \text{ so that} \int_{-\infty}^{\cot a} \frac{k}{1+w^2}\, dw = \frac{1}{2\pi} (\pi - a).$$

Thus $\frac{1}{2}\, g(w) = \frac{k}{1+w^2}$ and the result follows. This can be generalized as follows.

<u>Theorem 5</u>: Given the hypothesis of Theorem 3, let $w = x/y$ and $h(u)$, the density function for $u = \operatorname{arccot} w$, vanish outside the interval $[0,\pi]$. Then the density function f for x satisfies the integral equation

$$\int_0^{\infty} f(y)\, f(wy)y\, dy = \frac{k\, h(\cot^{-1} w)}{1+w^2}\ , \ k \text{ a constant.}$$

Proof: Following the model and notation of the last proof we have

$$\int_{-\infty}^{\cot a} \frac{1}{2}\, g(w)dw = \frac{1}{2}\left[1 - H(a)\right]\ , \quad 0 \leq a \leq \pi.$$

We need a function ℓ such that $\ell(\cot a)\csc^2 a = \frac{1}{2}\, h(a)$,

i.e. $\ell(w) = \frac{1}{2}\, \frac{h(\cot^{-1} w)}{1+w^2}$. Consider $\frac{1}{2}\int_{-\infty}^{\cot a} \frac{h(\cot^{-1} w)}{1+w^2}\, dw.$

Letting $y = \cot^{-1} w$ this becomes

$$-\frac{1}{2}\int_{\pi}^{a} h(y)dy = \frac{1}{2}\int_a^{\pi} h(y)dy = \frac{1}{2}\left[H(\pi) - H(a)\right] = \frac{1}{2}\left[1 - H(a)\right]$$

and the result follows.

3. Some Comments on the General Problem

It has already been noted that the density function for the quotient of two random variables x_1 and x_2 with density functions $f(x_1)$ and $g(x_2)$ respectively has the general form

$$\varphi(y_1) = \int_{-\infty}^{\infty} |y_2| f(y_1 y_2)\, g(y_2) dy_2$$

where $y_1 = x_1/x_2$ and $y_2 = x_2$. Writing the kernel $k(y_1,y_2) = |y_2| f(y_1 y_2)$ this has the form of a linear integral equation of the first kind and in operator notation can be formally written as $Kg = \varphi$. It appears somewhat difficult to determine the proper domain and range for this operator so as to apply directly to probability functions. The set of density functions in $C[a,b]$ or $L^2[a,b]$ for example, does not form a linear space. If the equations could be modified to consider distribution functions, addition can be defined as $F + G = F * G$ where $*$ means convolution but scalar "multiplication" appears to move one off the intersection of the unit ball with the positive cone in the Banach space under consideration.

Apparently then, the analysis should be done on some other space (as far as solutions are concerned) and a second analysis done to determine whether the solution or which of the solutions are density functions. If, for example, the kernel k vanishes outside the square $[a,b] \times [a,b] = E$ and $k \in L^2(E)$, then the operator K acting on L^2 is compact (with range in L^2). The theory of compact operators could possibly be extended so as to

apply to equations of the type needed here. To this end it should be mentioned that S. Birnbaum, at the University of Colorado and Martin-Denver, is presently developing a theory for pairs of operators acting between Banach spaces. This theory appears to have some applications in this area. We give a brief discussion here as to the type of results to expect. They will be called pretheorems.

We will be considering the spaces $L = L^p\,[0,1]$ and $C = C\,[0,1]$, $1 < p < \infty$, and an integral equation of the above form $Kf = g$, $K: L \longrightarrow C$. With proper restrictions on the kernel $k(s,t)$ determining K, K will be a continuous operator $\left[\text{viz. } k(s,t) \text{ continuous in } s \text{ for every } t \text{ and } \int_0^1 \left| k(s,t) \right|^{p'} dt < \infty \text{ for every } s \in [0,1], \text{ where } \frac{1}{p} + \frac{1}{p'} = 1 \right]$. Let $R : C \longrightarrow L^p$ be an imbedding, i.e., for $y \in C$, $Ry = [y] \in L^p$, ($[\ ]$ denotes an equivalence class). Then R is continuous and 1-1 and hence $R^{-1} = S$ exists as a closed operator $S : L^p \longrightarrow C$ and the domain of S is the set $\left\{ [y] \mid \exists\, y \in [y],\ y \in C \right\}$. Using the above theory for the pair (S,K) the following can then be proved.

<u>Pretheorem</u>: If there is λ (complex) such that $(S - \lambda K)$ is 1-1, the range of $(S - \lambda K)$ is C, and such that $\| S(S - \lambda K)^{-1} \| < 1$ then K restricted to the domain of S is 1-1, has range C, and has a continuous inverse defined everywhere on C.

It should be noted that among other things this is a uniqueness theorem. Applied to the problem considered in this paper the solution would have to be examined to determine whether or not it is a density function. From the form of the solution, however, it appears that this may be a difficult

problem but it has not yet been investigated. It also appears promising, using the new theory, that it will be possible to characterize the null space of $\mathbf{K}$ so that something can be said when multiple solutions are involved. Although the above result was stated for the unit interval, it can be extended to more general settings.

II. NOTES ON THE PRODUCT OF TWO NORMALLY DISTRIBUTED RANDOM VARIABLES.

1. Introduction.

Let X_1 and X_2 follow a normal bivariate probability density function, p.d.f., with expected values μ_1, μ_2, standard deviations, σ_1, σ_2, and coefficient of correlation, ρ. Several forms of random variable products may be considered; two of which are the normalized product $Z = (X_1 - \mu_1) \cdot (X_2 - \mu_2)/\sigma_1 \sigma_2$ and the product $Z = X_1 X_2/\sigma_1 \sigma_2$. The latter presents far greater application in that families of normal random variables X_1/σ_1 and X_2/σ_2 may be characterized by the statistics $v_i = \mu_i/\sigma_i$, $i = 1, 2$. These, of course, are the reciprocals of the respective coefficients of variation.

The joint p.d.f. of the normal random variables X_1/σ_1 and X_2/σ_2 is

$$f\left(\frac{x_1}{\sigma_1}, \frac{x_2}{\sigma_2}\right) = \frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1}{\sigma_1} - v_1\right)^2 - 2\rho\left(\frac{x_1}{\sigma_1} - v_1\right)\left(\frac{x_2}{\sigma_2} - v_2\right) + \left(\frac{x_2}{\sigma_2} - v_2\right)^2\right]\right\}}{2\pi\sqrt{1-\rho^2}} .$$

With the transformation $W = X_1/\sigma_1$, $Z = X_1 X_2/\sigma_1 \sigma_2$, the marginal p.d.f. of Z may be derived from

$$\varphi(z) = \int_{-\infty}^{\infty} \frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left[(w - v_1)^2 - 2\rho\,(w - v_1)(\frac{z}{w} - v_2) + (\frac{z}{w} - v_2)^2\right]\right\}}{2\pi\sqrt{1-\rho^2}\;|w|}\, dw \,. \tag{II-1}$$

The p.d.f. (II-1) may be expressed as $\varphi(z) = I_1(z) - I_2(z)$ where

$$I_1(z) = \int_0^{\infty} \frac{\exp\left\{ - \frac{1}{2(1-\rho^2)} \left[(w - v_1)^2 - 2\rho (w - v_1)(\frac{z}{w} - v_2) + (\frac{z}{w} - v_2)^2 \right] \right\}}{2\pi \sqrt{1-\rho^2}\; w} \, dw$$

and $I_2(z)$ is the same function defined on $(-\infty, o)$. After the substitution $w = -w$ into $I_2(z)$, the marginal p.d.f. $\varphi(z)$ may be expressed as

$$\varphi(z) = \frac{1}{2\pi \sqrt{1-\rho^2}} \int_0^{\infty} \frac{\exp\left\{ - \frac{1}{2(1-\rho^2)} \left[(w-v_1)^2 - 2\rho (w-v_1)(\frac{z}{w} - v_2) + (\frac{z}{w} - v_2)^2 \right] \right\}}{w} +$$

$$+ \frac{\exp\left\{ - \frac{1}{2(1-\rho^2)} \left[(w+v_1)^2 - 2\rho (w+v_1)(\frac{z}{w} + v_2) + (\frac{z}{w} + v_2)^2 \right] \right\}}{w} \, dw \,;$$

(II-2)

and by expanding these exponents and regrouping terms, (II-2) becomes

$$\varphi(z) = \frac{\exp\left\{ - \frac{1}{2(1-\rho^2)} \left[v_1^2 + v_2^2 - 2\rho \left[z + v_1 v_2 \right] \right] \right\}}{\pi \sqrt{1-\rho^2}} \int_0^{\infty} \frac{e^{- \frac{1}{2(1-\rho^2)} \left[w^2 + \frac{z^2}{w^2} \right]}}{w} \cdot$$

$$\cdot \cosh \left[\frac{w^2 (\rho v_2 - v_1) + z(\rho v_1 - v_2)}{w(1-\rho^2)} \right] dw \,. \qquad \text{(II-3)}$$

Several special cases may now be examined. When $v_1 = v_2 = 0$, the p.d.f. of the "normalized" product is obtained. The p.d.f. of Z in this case is[1]

$$\varphi(z) = \frac{e^{\frac{\rho}{1-\rho^2} z}}{\pi \sqrt{1-\rho^2}} K_o\left(\frac{z}{1-\rho^2}\right), \tag{II-4}$$

where $K_o(\cdot)$ is a modified Bessel function of the second kind of zero order possessing a singularity at $z = 0$. The product of two independent "normalized" variables by (II-4) reduces to

$$\varphi(z) = \frac{1}{\pi} K_o(z) , \tag{II-5}$$

a result shown in [5] and [6].

The non-central product $Z = X_1 X_2/\sigma_1 \sigma_2$ in which each variable is characterized by its respective reciprocal of the coefficient of variation, $v_i \neq 0$, has undergone extensive study. As yet, however, no satisfactory method of obtaining numerical results for the cumulative distribution function of Z has been derived for all parameters values of ρ , v_1, and v_2. The analysis by C. C. Craig [5], [6], is perhaps the most notable concerning this product.

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1 J. Wishart and M.S. Bartlett: The Distribution of Second Order Moments Statistics in a Normal System; Proceedings of the Cambridge Philosophical Society, Vol. 28, 1932, pp. 455-459.

In an effort to simplify any numerical calculation, Craig reformulated (II-3) as an infinite series. The cosh function in (II-3) may be expanded so that it is possible to write $\varphi(z) = I_1(z) - I_2(z)$ where

$$I_1(z) = \frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left[v_1^2 + v_2^2 - 2\rho\left[z + v_1 v_2\right]\right]\right\}}{2\pi\sqrt{1-\rho^2}} \int_0^\infty \exp\left\{-\frac{1}{2(1-\rho^2)}\left[w^2 + \frac{z^2}{w^2}\right] + \right.$$

$$\left. + \frac{1}{(1-\rho^2)}\left[(\rho v_2 - v_1)w + (\rho v_1 - v_2)\frac{z}{w}\right]\right\}\frac{dw}{w} \qquad \text{(II-6)}$$

and $I_2(z)$ is the integral of the same function over the interval $(-\infty, o)$. The infinite series expression is derived by substituting

$$\frac{w}{\sqrt{1-\rho^2}} = u \quad \text{and} \quad \frac{z}{1-\rho^2} = y \text{ into } I_1(z) \text{ and } I_2(z).$$

Under this transformation, $I_1(z)$ becomes

$$I_1(y) = \frac{\sqrt{1-\rho^2}}{2\Pi} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[v_1^2 + v_2^2 - 2\rho\left[y + v_1 v_2\right]\right]\right\} \int_0^\infty e^{-\frac{1}{2}\left[u^2 + \frac{y^2}{u^2}\right]}.$$

$$\cdot \exp\left\{\frac{(\rho v_2 - v_1)}{\sqrt{1-\rho^2}}u + \frac{(\rho v_1 - v_2)}{\sqrt{1-\rho^2}}\frac{y}{u}\right\}\frac{du}{u}. \qquad \text{(II-7)}$$

The term $\dfrac{\exp\left\{\frac{(\rho v_2 - v_1)}{\sqrt{1-\rho^2}}u + \frac{(\rho v_1 - v_2)}{\sqrt{1-\rho^2}}\frac{y}{u}\right\}}{u}$ may be expanded in a Laurent

series in powers of u for all u, $u \neq 0$. This expression is simplified to some extent by substituting

$$\frac{(\rho\, v_2 - v_1)}{\sqrt{1-\rho^2}} = R_1 \quad \text{and} \quad \frac{(\rho\, v_1 - v_2)}{\sqrt{1-\rho^2}} = R_2.$$

In the expansion, the coefficient of u^{r-1}, $r \geq 1$, is $\frac{R_1^{\,r}}{r\,!} \sum_r (R_1R_2y)$ in which $\sum_r (\cdot)$, the confluent hypergeometric function of order r, is[2]

$$\sum_r (R_1R_2y) = 1 + \frac{R_1R_2y}{r+1} + \frac{(R_1R_2y)^2}{(r+2)^{(2)}2!} + \frac{(R_1R_2y)^3}{(r+3)^{(3)}3!} + \ldots,$$

with $(r+k)^{(k)} = (r+k)(r+k-1)\ldots(r+1)$.

By this expansion and a similar expansion for $I_2(y)$, the p.d.f. of $Y = X_1X_2/\sigma_1\,\sigma_2\,(1-\rho^2)$ may be expanded in an infinite series involving confluent hypergeometric functions and powers y, v_1, and v_2. This series is

$$\begin{aligned}\varphi(y) = \frac{\sqrt{1-\rho^2}}{\pi} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[v_1^{\,2} + v_2^{\,2} - 2\rho\left[y + v_1v_2\right]\right]\right\} \Bigg[& \sum_0 (R_1R_2y)K_0(y) + \\ & + (R_1^{\,2} + R_2^{\,2})\frac{|y|}{2!}\sum_2 (R_1R_2y)K_1(y) + (R_1^{\,4} + R_2^{\,4})\frac{y^2}{4!}\sum_4 (R_1R_2y)K_2(y) + \\ & + (R_1^{\,6} + R_2^{\,6})\frac{|y|^3}{6!}\sum_6 (R_1R_2y)K_3(y) + \ldots \Bigg]\end{aligned} \tag{II-8}$$

where: $K_i(y)$ = the Bessel function of the second kind of the i^{th} order

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[2] These functions are discussed in detail in Whitaker, E.T., and G.N. Watson, A Course in Modern Analysis, Cambridge University Press, Cambridge, 1958.

and $\sum_j (R_1R_2y) = \frac{j!}{R_1{}^j}\left(\frac{R_1}{R_2y}\right)^{\frac{j}{2}} I_j\left(2\sqrt{v_1v_2\,y}\right)$ in which $I_j(\cdot)$ is the

Bessel function of the first kind of the j^{th} order.

When $v_1 = v_2 = \rho = 0$, the p.d.f. of $Z = X_1X_2/\sigma_1\,\sigma_2$ is the simple Bessel function expressed by (II-5).

Craig's result has unfortunately proved to be of little use computationally for it may be shown that for large v_1 and v_2 (a frequent occurrence in engineering studies) the series expansion converges very slowly; in fact, for v_1 and v_2 as small as 2, the expansion is unwieldy.

L.A. Aroian[1],[2] took up the problem of convergence in Craig's series expansion. Using Craig's notation, he showed that as v_1 and $v_2 \longrightarrow \infty$ the p.d.f. of $Z = X_1X_2/\sigma_1\,\sigma_2$ approaches the normal p.d.f. In addition, he demonstrated that the Type III function and the Gram-Charlier type A series afford excellent approximations to the distribution of Z when $\rho = 0$.

The characteristic function of $Z = X_1X_2/\sigma_1\,\sigma_2$ is

$$\psi_z(t) = \frac{\exp\left\{-\dfrac{(v_1{}^2+v_2{}^2-2\rho\,v_1v_2)t^2+2v_1v_2it}{2\left[1-(1+\rho)it\right]\left[1+(1-\rho)it\right]}\right\}}{\sqrt{\left[1-(1+\rho)it\right]\left[1+(1-\rho)it\right]}} \quad . \tag{II-9}$$

Using properties of this function, it is possible to show that $E\left[z\right] = \bar{z} = v_1v_2+\rho$ and the standard deviation is $\sigma_z = \sqrt{v_1{}^2+v_2{}^2+2\rho\,v_1v_2+1+\rho^2}$.

Aroian[1]proved the following statements :

1) The p.d.f. of Z approaches the normal p.d.f. with mean $\bar{z}$ and variance σ_z^2 as v_1 and $v_2 \longrightarrow \infty$ (or $-\infty$) in any manner whatsoever, provided $-1 + \epsilon < \rho < 1$, $\epsilon > 0$.

2) The p.d.f. of Z approaches the normal p.d.f. with mean $\bar{z}$ and variance σ_z^2 if $v_1 \longrightarrow \infty$ and $v_2 \longrightarrow -\infty$, provided $-1 \leq \rho < 1 - \epsilon$, $\epsilon > 0$.

and 3) The p.d.f. of Z approaches a normal p.d.f. if v_1 remains constant and $v_2 \longrightarrow \infty$, $-1 + \epsilon < \rho \leq 1$, $\epsilon > 0$; or if v_1 remains constant and $v_2 \longrightarrow -\infty$ for $-1 \leq \rho < 1 - \epsilon$, $\epsilon > 0$.

2. Numerical Computation.

2.1 Integration of the Cumulative Distribution Function: The cumulative distribution function F(z) may be formulated directly by making use of the fact that if $\psi_z(t)$ is the characteristic function of random variable Z, then the c.d.f. of Z is given by

$$F(z) = .50 + \frac{1}{2\pi}\int_0^\infty \frac{\cos tz}{it}\left\{\psi(-t) - \psi(t)\right\} + \frac{\sin tz}{t}\left\{\psi(-t) + \psi(t)\right\} dt \; . \tag{II-10}$$

This relation has been proved in[8]and[9]. The advantages of this formula lie in the fact that a separate determination of F(0) need not be made and a double numerical integration is avoided. Aroian[2]used (II-10)

to obtain numerical results when $\rho = 0$. In this case $F(z)$ may be expressed as

$$F(z) = .50 + \frac{1}{2\Pi}\int_0^{\infty} \frac{\exp\left\{\dfrac{-(v_1^2 + v_2^2)t^2}{2(1+t^2)}\right\}}{t\sqrt{1+t^2}} \sin\left\{t\left[z - \frac{v_1 v_2}{1+t^2}\right]\right\} dt. \tag{II-11}$$

This expression was numerically integrated from 0 to t_1, t_1 to t_2, . . ., t_i to t_{i+1}, $i = 1, 2, \ldots$, where t_i are the zeros of $\sin\left\{t\left[z - \frac{v_1 v_2}{1+t^2}\right]\right\}$. Aroian's tables of this c.d.f. include combinations of v_1 and v_2 at intervals of 0.4, $0 \leq v_1 \leq 4$, $0 \leq v_2 < 4$. The values of z are given at intervals of 0.1 for $\mu_z \pm \sigma_z$; at 0.2 for $(\mu_z + \sigma_z)$ to $(\mu_z + 3\sigma_2)$ and for $(\mu_z - \sigma_z)$ to $(\mu_z - 3\sigma_z)$; at intervals of 0.4 for $(\mu_z + 3\sigma_z)$ to $(\mu_z + 4\sigma_z)$ and for $(\mu_z - 3\sigma_z)$ to $(\mu_z - 4\sigma_z)$ and in intervals of 0.8 to the extreme values $\mu_z \pm 7\sigma_z$.

In theory the c.d.f. of the correlated product may be derived from (II-10). However, the resulting expression is quite complicated. Its rather cumbersome nature hinders the derivation of a substantial quantity of numerical results using the type of "intermediate" computer dictated by the scope of this study. Using $\mu_z(t)$ as given by (II-9), the c.d.f., F(z), of the correlated product is

$$F(z) = .50 + \frac{1}{\pi}\int_0^{\infty} \frac{\exp\left\{-\frac{1}{2}\left[\dfrac{t^2(k_3+k_1k_3t^2+4\rho v_1 v_2)}{1+2t^2k_2 + t^4k_1^2}\right]\right\}}{t}\left[\sqrt{\frac{1+t^2k_1+ \quad 1+2t^2k_2+t^4k_1^2}{2(1+2t^2k_2+t^4k_1^2)}}\right. .$$

$$\cdot \sin\left\{ t\left[z + \frac{(k_3 t^2 \rho - v_1 v_2(1+t^2 k_1))}{(1+2t^2 k_2 + t^4 k_1{}^2)} \right] \right\} - \sqrt{\frac{-1-t^2 k_1 + \quad 1+2t^2 k_2 + t^4 k_1{}^2}{2(1+2t^2 k_2 + t^4 k_1{}^2)}} \cdot$$

$$\cdot \cos\left\{ t\left[z + \frac{(k_3 t^2 \rho - v_1 v_2(1+t^2 k_1))}{(1+2t^2 k_2 + t^4 k_1{}^2)} \right] \right\} dt\,, \qquad \text{(II-12)}$$

where: $k_1 = (1-\rho^2)$, $k_2 = (1+\rho^2)$, $k_3 = (v_1{}^2 + v_2{}^2 - 2\rho\, v_1 v_2)$.

In order to obtain the zeros of the sin and cos functions, it is necessary to solve the fifth-order polynomial representing the arguments of the trigonometric functions. Numerical integration from zero to zero of each of these functions may be accomplished in a number of ways since all derivatives of these trigonometric functions are bounded. The number of zeros of both functions is greatly increased however in comparison with (II-11). In addition a bound for the tail area in (II-12) is difficult to obtain. Due to these difficulties and the limitations of the available computer, no further consideration was given (II-12) as a method of generating a large volume of tabular results.

2.2. <u>Integration to Obtain the Probability Density Function</u>: An alternate approach to obtaining $F(z)$ is of course a double numerical integration of (II-2). A rearrangement of the exponents in this equation will allow $\phi(z)$ to be expressed as

$$\frac{\varphi(z)}{K(z)} = \int_0^\infty \frac{e^{-\frac{1}{2(1-\rho^2)}\left(\frac{z^2}{w^2} + \frac{2z}{w}(\rho\, v_1 - v_2)\right)}\; e^{-\frac{1}{2(1-\rho^2)}\left(w^2 + 2(\rho\, v_2 - v_1)w\right)}}{w} +$$

$$+ \frac{e^{-\frac{1}{2(1-\rho^2)}\left(\frac{z^2}{w^2} - \frac{2z}{w}(\rho\, v_1 - v_2)\right)}\; e^{-\frac{1}{2(1-\rho^2)}\left(w^2 - 2(\rho\, v_2 - v_1)w\right)}}{w}\, dw\,, \tag{II-13}$$

where $K(z) = \exp\left\{-\frac{1}{2(1-\rho^2)}\left[v_1^2 + v_2^2 - 2\rho\left[z + v_1 v_2\right]\right]\right\} / 2\pi\sqrt{1-\rho^2}$.

In turn, (II-13) may be expressed as the sum of the integrals of the two functions of the integrand. Thus, let

$$\frac{\varphi(z)}{K(z)} = \int_0^\infty I_1(w)dw + \int_0^\infty I_2(w)dw\,, \tag{II-14}$$

where $I_1(w)$ and $I_2(w)$ are the respective terms of (II-13).

In order to bound the tail areas of (II-14) by ϵ, $\epsilon \le 10^{-5}$ say, it is sufficient to require that

$$\int_{u_{L_1}}^\infty I_1(w)dw + \int_{u_{L_2}}^\infty I_2(w)dw \le \epsilon_1 + \epsilon_2 \le \epsilon\,, \tag{II-15}$$

where u_{L_1} and u_{L_2} are the upper limits of the numerical integrations of $I_1(w)$ and $I_2(w)$, respectively, and ϵ_1 and ϵ_2 are defined by (II-16) and

(II-18). It is possible to bound each of these integrals by the normal probability integral, i.e., it is possible, for $I_1(w)$ say, to write

$$\int_{u_{L_1}}^{\infty} I_1(w)dw = \int_{u_{L_1}}^{\infty} \frac{e^{-\frac{1}{2(1-\rho^2)}\left(\frac{z^2}{w^2} + \frac{2z}{w}(\rho v_1 - v_2)\right)} e^{-\frac{1}{2(1-\rho^2)}\left(w^2 + 2(\rho v_2 - v_1)w\right)}}{w} dw$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{u_{L_1}}^{\infty} e^{-\frac{w^2}{2}} dw = \epsilon_1 \qquad \text{(II-16)}$$

The signs of $z, (\rho v_1 - v_2), (\rho v_2 - v_1)$ may combine to produce either positive or negative terms in the exponents of (II-16). In an error analysis, the selection of an appropriate w_o to insure, for all $w > w_o$, that

$$\frac{\exp\left\{-\frac{1}{2}\left(\frac{z^2 \pm 2\left|z(\rho v_1 - v_2)\right|w}{w^2(1-\rho^2)}\right)\right\}}{w} \leq \frac{1}{2\pi} \quad \text{and}$$

$$\exp\left\{-\frac{1}{2}\left(\frac{w^2 \pm 2\left|\rho v_2 - v_1\right|w}{(1-\rho^2)}\right)\right\} \leq e^{-\frac{w^2}{2}} \qquad \text{(II-17)}$$

represents one method of satisfying (II-16). Similarly, the tail area of $I_2(w)$ may be bounded by

$$\int_{u_{L_2}}^{\infty} I_2(w)dw \le \frac{1}{\sqrt{2\pi}} \int_{u_{L_2}}^{\infty} e^{-\frac{w^2}{2}} dw = \epsilon_2 , \tag{II-18}$$

provided w_o is chosen so that

$$\frac{\exp\left\{ -\frac{1}{2}\left(\frac{z^2 \mp 2\,|z(\rho\, v_1 - v_2)|\,w}{w^2(1-\rho^2)} \right) \right\}}{w} \le \frac{1}{\sqrt{2\pi}} \text{ and}$$

$$\exp\left\{ -\frac{1}{2}\left(\frac{w^2 \mp 2\,|\rho\, v_2 - v_1|\,w}{(1-\rho^2)} \right) \right\} \le e^{-\frac{w^2}{2}} , \tag{II-19}$$

for all $w > w_o$.

For all $w_o > \sqrt{2\pi}$, the sets of inequalities (II-17) and (II-19) hold when the signs within the exponents of these two sets are positive. In this case, the upper limits u_{L_i} of numerical integration of $I_1(w)$ and $I_2(w)$ may be chosen as $T_1 = T_2 = 4.42$, respectively. In this case $\epsilon_1 = \epsilon_2$ and from an appropriate table each is computed to be less than 5×10^{-6}. The total tail area ϵ is then less than 10^{-5}.

The other "extreme" case arises when the pair of signs in either one of the sets of inequalities, say (II-17), is negative. The first inequality of (II-17) then requires that

$$w \exp\left\{ \frac{1}{2}\left(\frac{z^2 - 2\,|z(\rho\, v_1 - v_2)|\,w}{w^2(1-\rho^2)} \right) \right\} \ge \sqrt{2\pi} . \tag{II-20}$$

In most cases there will be two sets of values of w satisfying this inequality. An approximation to the least upper bound of the upper range of these values, $\mathscr{A}$, may be obtained by a numerical iteration of

$$2(1-\rho^2)\log\left\{\frac{w}{\sqrt{2\pi}}\right\} w^2 - \left|2\, z(\rho\, v_1 - v_2)\right| w + z^2 > 0. \qquad \text{(II-21)}$$

The set of w satisfying the second inequality of (II-17) is easily shown to be $w > 2\left|\rho\, v_2 - v_1\right|/\rho^2$. In <u>every</u> case the tail area of (II-16) may be bounded by $\frac{1}{\sqrt{2\pi}}\int_{T_1=4.42}^{\infty} e^{-\frac{w^2}{2}}\, dw = \epsilon_1 \leq 5 \times 10^{-6}$ provided the numerical integration is performed over the interval $(0, u_L)$ where

$$u_L = \max\left\{ \mathscr{A},\ 2\left|\rho\, v_2 - v_1\right|/\rho^2,\ T_1 = 4.42 \right\}. \qquad \text{(II-22)}$$

The upper limit given by (II-21) is quite obviously an upper limit of integration for $I_2(w)$ by the same argument. Thus, the tail area estimate of (II-15) may be restricted by $\epsilon < \epsilon_1 + \epsilon_2 < 10^{-5}$ provided the upper limits of the numerical integration for both $I_1(w)$ and $I_2(w)$ are determined by (II-22).

2.3 <u>Methods of Numerical Integration</u>: Several formulas such as Weddel's formula, the trapezoidal rule, the Gregory-Newton formula, and the simple rectangular formula [2], [5] have been suggested for the numerical

integration of

$$\frac{\phi(z)}{K(z)} = \int_0^{u_{L_1}} I_1(w)dw + \int_0^{u_{L_2}} I_2(w)dw . \qquad \text{(II-23)}$$

The magnitude of the error bounds for these and most other numerical integration methods depends directly or indirectly on values of a given derivative of the integrand within the interval of integration. The first derivatives of $I_1(w)$ and $I_2(w)$ may be written in the form of a rational function,

$$\frac{d}{dw} I_i(w) = J_i(w) \left[\frac{z^2(-1)^{i+1} z(\rho v_1 - v_2)w - (1-\rho^2)w^2(-1)^{i+2}(\rho v_2 - v_1)w^3 - w^4}{(1-\rho^2)w^4} \right] \qquad \text{(II-24)}$$

where $J_i(w)$, $i=1, 2$, are the exponential functions of the $I_i(w)$. Alternately, (II-24) may be expressed as

$$\frac{d}{dw} I_1(w) = \frac{J_i(w)}{(1-\rho^2)\, w^{n=4}} P(w) , \qquad \text{(II-25)}$$

where $P(w)$ represents the polynomial of (II-24).

All derivatives of $I_i(w)$ may be expressed in the form (II-25) with the order n of the polynomial $P(w)$ increasing accordingly. The terms of $P(w)$ in the second and higher-order derivatives are various powers and cross-products of the parameters z, $(\rho v_2 - v_1)$, $(\rho v_1 - v_2)$, and $(1-\rho^2)$.

The error estimate for a given method of numerical integration is a function of the maximum value within the interval of integration of the derivative associated with that method. For example, the error bound of the trapezoidal rule is a function of the second derivative, Weddel's formula involves the sixth derivative, etc. The actual error, of course, may be much smaller than indicated by the error bound.

In order to calculate the maximum values of the given n^{th} derivative within the interval of integration, it is necessary to derive and solve for the roots of the polynomial of the $(n+1)^{st}$ derivative. This task becomes increasingly difficult to do analytically as the order of $P(w)$ increases. A numerical iteration method must be used to solve for the roots of $P(w)$ in the higher-order derivatives.

The values of each derivative are functions of the values of the parameters and their signs. Thus, for appropriate sets of parameter values, the derivatives are large in the neighborhood of $w = 0$, or more generally in the interval $(0,1)$. As yet, no method of characterizing the derivatives within the interval of integration as functions of the parameter values has proved satisfactory. Because of these difficulties, other numerical methods which are not functionally dependent of the derivatives are believed to be more expedient for this problem.

The c.d.f.'s of z appearing in Appendix B were obtained using a double numerical integration of (II-23) by the simple upper and lower sum rectangular formula. In order to obtain $\phi(z)$, the real roots, R_1, R_2 . . ., of $P(w)$

for the first derivative of both $I_1(w)$ and $I_2(w)$, (II-24), were estimated by numerical methods to within five significant digits. For each integrand $I_i(w)$, an upper and lower sum, U_S and L_S, were computed in the intervals $(0$ to $R_1)$, ..., $(R_j$ to $u_{L_i})$ with a normal increment $\Delta w = 0.01$. The increment was reduced as necessary to insure that $U_S - L_S \leq 10^{-5}$ in all intervals. In many cases, the functions $I_i(w)$ are quite steep in the interval $(0, R_1)$ with $R_1 << 1$. In these cases, very small Δw's were required to obtain a satisfactory estimate in this interval.

As compared with other numerical methods, the rectangular formula provides greater accuracy but generally requires a much larger number of computer calculations. The actual computer time required to obtain $F(z)$ is dependent upon the shapes of $I_1(w)$, $I_2(w)$, and $\varphi(z)$. The "average" computer time required to approximate the double integration within the desired accuracy was approximately 58 minutes[3]. The computer time required for this integration program can be reduced to approximately 1.25 minutes using a high-speed computer such as an IBM 7094[4]. Considering both computer costs and the time required to generate the desired volume of tabular results, the use of an IBM 7094 or its equivalent is recommended for future work. The cost of using a computer on this scale prohibited its use in the investigation which was intended and funded as a "preliminary study".

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[3] Based on the use of an IBM 1620 computer.

[4] As estimated by members of the Martin Company Data Systems Division.

REFERENCES

1. Aroian, L. A., "The Probability Function of a Product of Two Normally Distributed Variables," Annals of Math. Stat. Vol. 18, 1947.

2. Aroian, L. A., Tables and Percentage Points of the Distribution Function of a Product, Hughes Aircraft Company, California, 1957.

3. Bartlett, M. and J. Wishart, "The Distribution of Second Order Moment Statistics in the Normal System," Proc. Camb. Phil. Soc., Vol. 28, 1932.

4. Craig, C. C., "The Frequency Function of X/Y," Annals of Math., Vol. 30, 1929.

5. Craig, C. C., "On the Frequency Function of XY," Annals of Math. Stat., Vol. 7, 1936.

6. Craig, C. C., "On the Frequency Distribution of the Quotient and the Product of Two Statistical Variables," American Math. Monthly, Vol. 49, 1942.

7. Huntington, E. V., "Frequency Distributions of Products and Quotients," Annals of Math. Stat., Vol. 10, 1939.

8. Gil-Pelaez, J., "Notes on the Inversion Theorem," Biometrika, Vol. 38, 1951.

9. Gurland, J., "Inversion Formulae for the Distribution of Ratios," Annals of Math. Stat., Vol. 19, 1948.

10. Pearson, K., Stauffer, S. A. and David, F. N., "Further Applications in Statistics of the $T_m'(x)$ Bessel Function," Biometrika, Vol. 24, 1932.

11. Cadwell, J. H., "The Bivariate Norm Integral," Biometrika, Vol. 38, 1951.

12. Tables of the Bivariate Normal Distribution Function and Related Functions, National Bureau of Standards, Vol. 50, Applied Math. Series, 1956.

13. Watson, G. N., Theory of Bessel Functions, Cambridge University Press, London, 1922.

14. Luke, Y. L., Integrals of Bessel Functions, McGraw-Hill Book Co., Inc., New York, 1962.

15. Tables of the Exponential Integral, National Bureau of Standards, Vol. 14, Applied Math. Series, 1951.

16. The Probability Integral, British Assoc. for the Advancement of Science, Math. Tables, Vol. VII, 1939.

17. Tables of Error Function and Its Derivative, National Bureau of Standards, Vol. 41, Applied Math. Series, 1954.

18. Titchmarsh, E. C., Theory of Fourier Integrals, Oxford University Press, 1948.

19. Springer and Thompson, "The Distribution of Products of Independent Random Variables," GM Defense Research Laboratories Report TR-64-46, August 1964.

20. Donahue, J., "Products and Quotients of Random Variables and Their Applications," ARL Report 64-115, July 1964.

21. Fox, C., "The Solution of An Integral Equation," Can. J. Math., Vol. XVI-3, 1964. pp 578-586.

22. Laha, R. G., "On the Laws of Cauchy and Gauss," Annals of Math. Stat., Vol. 30, 1959.

23. Laha, R. G., "On a Class of Distribution Functions Where the Quotient Follows the Cauchy Law," TAMS, Vol. 13, 1959.

24. Loeve, M., Probability Theory, Van Nostrand, New York, 1963.

25. Lukacs, E., Characteristic Functions, Hafner, New York, 1960.

26. Hirschman and Widder, The Convolution Transform, Princeton, 1955.

27. Cramér, H., Mathematical Methods of Statistics, Princeton, 1951.

28. Muskhelishvili, N., Singular Integral Equations, Translated by I. Radak, Noordhoff, 1953.

29. Bergstrom, H., Limit Theorems for Convolutions, Wiley, 1963.

30. Mikhlin, S., Linear Integral Equations, Hindustan Pub. Co. (India), 1960.

31. Mauldon, J., "Characterizing Properties of Statistical Distributions," Journal of Math., Oxford, Vol. 7, 1956.

32. Steck, G., "A Uniqueness Property Not Enjoyed by the Normal Distributions," Annals of Math. Stat., Vol. 29, 1958.

33. Laha, R. G., "An Example of a Nonnormal Distribution Where the Quotient Follows the Cauchy Law," Proc. of Nat. Aca. of Science, Vol. 44.

34. Goodspeed, F., "The Relation Between Functions Satisfying a Certain Integral Equation and General Watson Transforms," Can. J. Math., Vol. 2, 1950.

APPENDIX A

We give two results here that were obtained in a "side" investigation involving the quotients of random variables. One is probably not new but an easy proof is given.

Theorem: Let $f(z) = \sum_{i=o}^{n} b_i \; z^{n-i}$ be a complex polynomial, $b_o \neq 0$.

Then the zeros of $f(z)$ are in disk $|z| \leq 1 + \rho$ where $\rho = \sup \left| \frac{b_i}{b_o} \right|$, $i = 1, \ldots, n$.

Proof: Write $f(z) = b_o \; g(z)$ where

$$g(z) = z^n + \sum_{i=1}^{n} a_i \; z^{n-1}$$

and

$$a_i = \frac{b_i}{b_o} .$$

Then

$$|g(z)| = \left| z^n - \left(- \sum_{i=1}^{n} a_i \; z^{n-i} \right) \right| \geq |z^n| - \left| \sum_{i=1}^{n} a_i \; z^{n-i} \right| \geq |z^n| - \sum_{i=1}^{n} |a_i| \, |z|^{n-i} .$$

Let $\rho = \sup |a_i|$ and suppose $|z| > 1 + \rho$, then from the above

$$|g(z)| > |z|^n - \rho \sum_{i=1}^{n} |z|^{n-i} = |z|^n - \rho \left[\frac{|z|^n - 1}{|z| - 1} \right] = \frac{|z|^n \left(|z| - 1 - \rho \right) + \rho}{|z| - 1} .$$

But $|z| - 1 - \rho > 1 + \rho - 1 - \rho = 0$ and hence for $|z| > 1 + \rho$, $\quad |g(z)| > 0$.

Consequently $|f(z)| = |b_0|\,|g(z)| > 0$ for this range of z and the result follows.

The question arose in a study of mappings between various topological spaces, when the arbitrary union of closed sets is closed. As a partial answer we give the following. Notationally for the space Y, let $2^Y = \{ E \subset Y \mid E \text{ is closed and non-empty} \}$. A mapping f, from a space X into 2^Y is said to be <u>upper semi-continuous</u> in case $x_0 \in X$, U open in Y and $f(x_0) \subset U$ implies that there is an open set V in X, $x_0 \in V$, such that $x \in V$ implies that $f(x) \subset U$ the above to hold, of course, for each $x_0 \in X$.

Theorem: Let X be a compact, Hausdorff space, Y regular, and let f be an upper semi-continuous function from $X \longrightarrow 2^Y$. Then $\bigcup_{x \in X} f(x)$ is closed in Y.

Proof: Let $B = \bigcup_{x \in X} f(x)$ and assume $y \in \overline{B}$, $y \notin B$ where $\overline{B}$ denotes the closure of B in Y. Clearly then $y \notin f(x)$ for any x. This implies that for each x there are two disjoint open sets in Y, $W_{f(x)}$ and $\Theta_{yf(x)}$ such that $f(x) \subset W_{f(x)}$ and $y \in \Theta_{yf(x)}$, $\Theta_{yf(x)} \cap W_{f(x)} = \emptyset$. Since f is upper semi-continuous, for each $x \in X$ there is an open V_x in X, $x \in V_x$, such that $f(\overline{x}) \subset W_{f(x)}$ for each $\overline{x} \in V_x$. Hence $\{V_x\}$ is a cover for X, a compact space. Consequently a finite number will do, say $X = \bigcup_{i=1}^{n} V_{x_i}$. Then $y \notin Z = \bigcup_{i=1}^{n} W_{f(x_i)} \supset \bigcup_{x \in X} f(x) = B$. Now $\Theta_y = \bigcap_{i=1}^{n} \Theta_{yf(x_i)}$ is an open set containing y, and $\Theta_y \cap Z = \emptyset$ so that $\Theta_y \cap B = \emptyset$. But this is a contradiction since y was assumed to be in $\overline{B}$ and the theorem is proved.

APPENDIX B

TABLES OF THE PRODUCT OF TWO NORMALLY DISTRIBUTED RANDOM VARIABLES

1. The c.d.f.'s of the random variable $Z = X_1X_2/\sigma_1\sigma_2$ for various parameter values were obtained by a double numerical integration of (II-23). In this preliminary study, only positive parameters, ρ, v_1, v_2, were considered.

2. Tail Area Bounds. Denoting the p.d.f. of Z in the correlated case as $f(z, \rho > 0)$, it is easily shown that $f(z, \rho > 0) < \phi(z, \rho = 0)$ for $z < 0$. Thus, $\int_{-\infty}^{z_0} f(z, \rho > 0)\, dz < \int_{-\infty}^{z_0} \phi(z, \rho = 0)\, dz = \lambda_1$, for $z_0 < 0$.

The value z_0 may be chosen so that λ_1 is arbitrarily small*. In addition, it may be shown that $f(z, \rho > 0) < \phi\left(\max\left\{(x_1/\sigma_1)^2, (x_2/\sigma_2)^2\right\}\right)$ for all $z > z_1 > 0$. Here the symbol $\phi\left(\max\left\{(x_1/\sigma_1)^2, (x_2/\sigma_2)^2\right\}\right)$ denotes the p.d.f. of the square of the largest of the random variables x_1/σ_1 and x_2/σ_2. This random variable follows the χ^2 p.d.f. with one degree of freedom. Thus for some $z_1 > 0$, it follows that $\int_{z_1}^{\infty} f(z, \rho > 0) dz < \int_{z_1}^{\infty} \phi\left(\max\left\{(x_1/\sigma_1)^2, (x_2/\sigma_2)^2\right\}\right) d(\bullet) = \lambda_2$. The values z_0 and z_1 may be chosen so that the sum, θ, of the probabilities λ_1 and λ_2 as determined by their respective c.d.f.'s is arbitrarily small. The integral value I of $f(z, \rho)$ for positive ρ may be estimated for the neighborhood $(\Delta_1 < 0 < \Delta_2)$ containing the point of discontinuity, $z = 0$, from the relation

$$I = 1 - \left(\int_{z_0}^{\Delta_1} f(z, \rho > 0) dz + \int_{\Delta_2}^{z_1} f(z, \rho > 0) dz + \theta \right).$$

The estimate I may be accurately approximated simply by requiring that θ be made small.

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*This function has been tabulated by L. A. Aroian [2].

3. <u>Checks</u>. Tables B-I to B-III may be compared with Aroian's results [2]. Those data points noted with an asterisk vary by 10^{-5} with his results.

B - I.

Parameter Values;

$\rho = 0., V_1 = 0., V_2 = 0.4$

Z	f(z)	F(z)	Z	f(z)	F(z)
-7.60	.00014	.00011	.10	.82755	.60294
-6.80	00028	.00025	.20	.55167	.66551
-6.00	.00064	.00057	.30	.43129	.71327
-5.20	.00084	.00128	.40	.35191	.75177
-4.40	.00287	.00290	.50	.29379	.78369
-4.00	.00468	.00439	.60	.24888	.81053
-3.60	.00715	.00665	.70	.21301	.83344
-3.20	.01027	.01011	.80	.19370	.85313
-3.00	.01332	.01249	.90	.15935	.87018
-2.80	.01658	.01544	1.00	.13489	.88500
-2.60	.02067	.01912	1.20	.10770	.90931
-2.40	.02854	.03371	1.40	.08356	.92809
-2.20	.03240	.02946	1.60	.06560	.94273
-2.00	.04076	.03667	1.80	.05171	.95424
-1.80	.05171	.04576	2.00	.04076	.96333
-1.60	.06560	.05727	2.20	.03240	.97654
-1.40	.08356	.07191	2.40	.02584	.97629
-1.20	.10770	.09069	2.60	.02067	.98088
-1.00	.13489	.11500	2.80	.01658	.98456
- .90	.15925	.12982	3.00	.01332	.98751
- .80	.18370	.14687	3.20	.01027	.98989
- .70	.21301	.16656	3.60	.00715	.99335
- .60	.24888*	.18948	4.00	.00468	.99561
- .50	.29379	.21634	4.40	.00287	.99710
- .40	.35191	.24823	5.20	.00145	.99872
- .30	.43129	.28672	6.00	.00064	.99943
- .20	.55167	.33449	6.80	.00028	.99975
- .10	.82755	.39706	7.60	.00014	.99989
.00	∞	.50000			

B - II .

Parameter Values;

$\rho = 0.,\ V_1 = 0.4,\ V_2 = 0.4$

Z	f(z)	F(z)	Z	f(z)	F(z)
-8.00	.00005	.00004	.20	.54599	.61086
-7.20	.00011	.00010	.30	.43713	.65878
-6.40	.00026	.00022	.40	.36474	.69828
-5.60	.00063	.00053	.50	.31109	.73173
-4.80	.00129	.00124	.60	.26905	.76050
-4.40	.00216	.00192	.70	.23494	.78554
-4.00	.00337	.00297	.80	.20661	.80749
-3.60	.00530	.00462	.90	.18268	.82686
-3.20	.00777	.00721	1.00	.16221	.84403
-3.00	.01024	.00903	1.10	.14453	.85930
-2.80	.01292	.01131	1.20	.12915	.87293
-2.60	.01634	.01420	1.30	.11569	.88513
-2.40	.02070	.01785	1.40	.10135	.89607
-2.20	.02633	.02248	1.60	.08453	.91474
-2.00	.03359	.02838	1.80	.06874	.92989
-1.80	.04303	.03592	2.00	.05615	.94224
-1.60	.05543	.04556	2.20	.04600	.95234
-1.40	.07187	.05809	2.40	.03775	.96063
-1.20	.09398	.07434	2.60	.03106	.96744
-1.00	.11913	.09568	2.80	.02560	.97305
-0.90	.14244	.10884	3.00	.02118	.97768
-0.80	.16476	.12407	3.20	.01746	.98151
-0.70	.19295	.14180	3.40	.01444	.98467
-0.60	.22646	.16256	3.60	.01148	.98728
-0.50	.26920	.18709	4.00	.00834	.99124
-0.40	.32465	.21640*	4.40	.00574	.99396
-0.30	.40048	.25204	4.80	.00370	.99583
-0.20	.51533	.29650	5.60	.00201	.99801
-0.10	.77595	.35509	6.40	.00095	.99905
0.00	∞	.45169	7.20	.00045	.99955
.10	.79584	.54958	8.00	.00028	.99978

B-III.

Parameter Values;

$\rho = 0.$, $V_1 = 0.4$, $V_2 = 1.2$

Z	f(z)	F(z)	Z	f(z)	F(z)
-11.00	.00001	.00001	.30	.36887	.53983
-10.00	.00002	.00002	.40	.33000	.56451
- 9.40	.00004	.00004	.50	.29967	.59584
- 8.60	.00009	.00009	.60	.27002	.63444
- 7.80	.00019	.00021	.70	.25287	.65074
- 7.00	.00040	.00042	.80	.23380	.67502
- 6.20	.00069	.00084	.90	.21672	.69750
- 5.80	.00114	.00121	1.00	.20116	.71846
- 5.40	.00166	.00176	1.10	.18775	.73775
- 5.00	.00241	.00255	1.20	.17418	.75579
- 4.60	.00350	.00369	1.30	.16223	.77259
- 4.20	.00459	.00535	1.40	.15116	.78824
- 4.00	.00581	.00644	1.50	.14089	.80282
- 3.80	.00725	.00775	1.60	.13137	.81642
- 3.60	.00827	.00934	1.70	.11999	.82909
- 3.40	.01059	.01126	1.80	.11170	.84091
- 3.20	.01279	.01358	1.90	.10649	.85195
- 3.00	.01548	.01638	2.00	.09930	.86221
- 2.80	.01874	.01977	2.10	.09258	.87180
- 2.60	.02272	.02388	2.20	.08489	.88073
- 2.40	.02759	.02886	2.40	.07517	.89682
- 2.20	.03856	.03492	2.60	.06527	.91080
- 2.00	.04091	.04229	2.80	.05666	.92293
- 1.80	.05002	.05129	3.00	.04914	.93345
- 1.60	.06137	.06229	3.20	.04256	.94257
- 1.40	.07666	.07583	3.40	.03685	.95048
- 1.20	.09096	.09256	3.60	.03244	.95732
- 1.10	.10414	.10239	3.80	.02758	.96323
- 1.00	.11659	.11338	4.00	.02383	.96835
- .90	.13589	.12570	4.20	.02058	.97277
- .80	.14748	.13955	4.40	.01776	.97658
- .70	.16702	.15519	4.60	.01531	.97987
- .60	.19030	.17295	4.80	.01320	.98271
- .50	.21865	.19325	5.00	.01137	.98515
- .40	.25425	.21669	5.20	.00974	.98726
- .30	.30128	.24411	5.40	.00881	.98907
- .20	.36983	.27694	6.20	.00594	.99197
- .10	.51705	.31807	7.00	.00263	.99411
- .00	∞	.38035	7.80	.00141	.99569
.10	.55191	.44480	8.60	.00075	.99685
.20	.42497	.49073	9.40	.00040	.99911

B- IV.

Parameter Values;

$\rho = 0.$, $V_1 = 0.5$, $V_2 = 0.5$

Z	f(z)	F(z)	Z	f(z)	F(z)
-9.60	.00001	.00000	.10	.68106	.47012
-8.80	.00002	.00001	.20	.51947	.53312
-8.00	.00004	.00004	.40	.36322	.61139
-7.20	.00010	.00009	.80	.21768	.72757
-6.40	.00023	.00022	1.20	.14230	.79956
-5.60	.00054	.00053	1.60	.09621	.84726
-4.80	.00128	.00126	2.00	.08589	.88368
-4.00	.00311	.00302	2.40	.06614	.91409
-3.60	.00438	.00469	2.80	.04589	.93650
-3.20	.00769	.00777	3.20	.02241	.95116
-2.80	.01221	.01108	3.60	.01571	.95778
-2.40	.01954	.01743	4.00	.01103	.96313
-2.00	.03164	.02765	4.80	.00545	.96815
-1.60	.05123	.04424	5.60	.00269	.97206
-1.20	.08806	.07210	6.40	.00133	.97467
- .80	.15568	.12087			
- .40	.30423	.21286			
- .20	.47388	.28465			
- .10	.64993	.34084			
.00	∞	.42666			

B- V.

Parameter Values;

$\rho = 0.3, \quad V_1 = 0.15, \quad V_2 = 0.4$

Z	f(z)	F(z)	Z	f(z)	F(z)
-6.84	.00001	.00002	.09	.80128	.45604
-6.12	.00003	.00006	.18	.55214	.51638
-5.40	.00008	.00010	.27	.44570	.56189
-4.68	.00025	.00014	.36	.37549	.59884
-3.96	.00065	.00041	.45	.32367	.62030
-3.60	.00122	.00138	.54	.28311	.65760
-3.24	.00212	.00272	.63	.25018	.68179
-2.88	.00346	.00373	.72	.22277	.70237
-2.70	.00479	.00537	.81	.19953	.72189
-2.52	.00635	.00673	.90	.17439	.73872
-2.34	.00844	.00773	1.08	[illegible]845	75761
-2.16	.01243	.00950	1.26	.12278	.79271
-1.98	.01504	.01237	1.44	.10276	.81239
-1.80	.02017	.01523	1.62	.08636	.82975
-1.62	.02729	.01953	1.80	.07257	.85317
-1.44	.03690	.02525	1.98	.06149	.85597
-1.26	.05012	.03309	2.16	.05229	.86603
-1.08	06887	.04382	2.34	.04460	.87371
- .90	.09195	.55739	2.52	.03813	.88228
- .81	.11209	.06149	2.70	.03267	.88866
- .72	.13350	.07762	2.88	.02684	.89401
- .63	.15984	.09082	3.24	.02125	.89761
- .54	.19283	.10637	3.60	.01580	.90933
- .45	.23503	.12593	3.96	.01101	.91415
- .36	.29068	.14957	4.68	.00723	.91744
- .27	.36784	.17924	5.40	.00410	.92039
- .18	.48580	.21765	6.12	.00234	.92286
- .09	.75244	.27337	6.84	.00154	.92831
.00	∞	.36359			

B- VI.

Parameter Values;

$\rho = 0.3, \quad V_1 = 0.55, \quad V_2 = 0.55$

Z	f(z)	F(z)	Z	f(z)	F(z)
-7.18	.00000	.00000	.09	.72204	.37697
-6.46	.00000	.00000	.18	.51147	.44612
-5.74	.00003	.00000	.27	.42280	.48066
-5.02	.00009	.00001	.36	.36426	.52420
-4.30	.00024	.00002	.45	.32079	.56513
-3.94	.00046	.00017	.54	.28645	.58245
-3.59	.00082	.00039	.63	.25827	.60697
-3.23	.00147	.00079	.72	.23451	.62914
-2.87	.00245	.00134	.80	.21409	.65110
-2.69	.00344	.00247	.89	.19629	.66780
-2.51	.00463	.00302	.98	.18058	.67032
-2.33	.00624	.00391	1.07	.16661	.68595
-2.15	.00844	.00490	1.16	.15410	.70038
-1.97	.01144	.00669	1.25	.13939	.71359
-1.79	.01556	.00811	1.43	.12395	.72865
-1.61	.02125	.01243	1.61	.10746	.74938
-1.43	.02919	.01696	1.79	.09358	.77621
-1.25	.04034	.02323	1.97	.08172	.79198
-1.07	.05624	.03190	2.15	.07150	.80578
- .89	.07601	.04382	2.33	.06272	.81786
- .80	.09384	.05146	2.51	.05511	.82837
- .72	.11207	.06073	2.69	.04860	.83779
- .63	.13552	.07187	2.87	.04271	.84601
- .54	.16422	.08436	3.24	.03093	.85926
- .45	.20157	.10181	3.59	.02062	.86854
- .36	.25100	.13219	4.30	.01511	.88016
- .27	.31969	.14787	5.02	.01060	.89066
- .18	.42475	.18137	5.74	.00652	.89682
- .09	.66035	.23020	6.46	.00402	.90061
.00	∞	.29747	7.18	.00320	.90322

B- VII.

Parameter Values;

$\rho = 0.3, \quad V_1 = 0.7, \quad V_2 = 0.7$

Z	f(z)	F(z)	Z	f(z)	F(z)
-8.60	.00000	.00000	- .18	.37439	.15722
-7.90	.00000	.00000	- .09	.53018	.19792
-7.20	.00000	.00001	.00	∞	.26839
-6.45	.00000	.00002	.09	.59228	.34165
-5.75	.00002	.00003	.18	.46645	.38929
-5.00	.00007	.00004	.36	.34770	.46256
-4.30	.00023	.00006	.72	.23683	.56770
-3.60	.00072	.00009	1.08	.17597	.64208
-3.25	.00129	.00437	1.44	.13521	.69809
-2.88	.00232	.00727	1.80	.13720	.74721
-2.52	.00419	.00844	2.16	.12008	.79343
-2.16	.00763	.01057	2.52	.09470	.83107
-1.80	.01405	.01447	2.88	.05256	.85847
-1.44	.02586	.02165	3.25	.04187	.87560
-1.08	.05052	.03540	3.60	.03343	.88908
- .72	.10151	.06277	4.30	.02135	.89105
- .36	.22545	.12163	5.00	.01363	.89214
			5.75	.00871	.90100

B- VIII.

Parameter Values;

$\varrho = 0.45, \quad V_1 = 0.2, \quad V_2 = 0.45$

Z	f(z)	F(z)	Z	f(z)	F(z)
-6.10	.00000	.00000	.08	.76179	.44780
-5.46	.00001	.00001	.16	.53068	.49950
-4.82	.00004	.00002	.24	.43354	.53807
-4.18	.00023	.00010	.32	.36966	.57175
-3.52	.00036	.00029	.40	.32249	.59788
-3.20	.00070	.00045	.48	.28548	.62220
-2.88	.00129	.00078	.56	.25533	.64383
-2.56	.00221	.00134	.64	.23009	.66325
-2.40	.00313	.00185	.72	.20858	.68079
-2.24	.00426	.00236	.80	.18450	.69652
-2.08	.00580	.00316	.96	.16087	.70459
-1.92	.00874	.00432	1.12	.13628	.71315
-1.76	.01084	.00589	1.28	.11684	.73340
-1.60	.01489	.00795	1.44	.10057	.75080
-1.44	.02063	.01079	1.60	.08656	.76577
-1.28	.02858	.01473	1.76	.07514	.81347
-1.12	.03975	.02019	1.92	.06544	.82472
- .96	.05595	.02785	2.08	.05717	.83453
- .80	.07652	.03845	2.24	.05007	.84310
- .72	.09441	.04529	2.40	.04394	.85063
- .64	.11380	.05362	2.56	.03698	.85710
- .56	.13790	.06368	2.88	.03071	.85795
- .48	.16837	.07593	3.20	.02397	.85806
- .40	.20769	.09098	3.52	.01752	.85972
- .32	.25997	.10968	4.18	.01266	.85997
- .24	.33295	.13349	4.82	.00791	.86003
- .16	.44504	.16452	5.46	.00498	.86117
- .08	.69762	.21023	6.10	.00361	.86203
.00	∞	.31104			

B- IX.

Parameter Values;

$\rho = 0.45, \quad V_1 = 0.8, \quad V_2 = 0.8$

Z	f(z)	F(z)	Z	f(z)	F(z)
-7.75	.00000	.00000	- .08	.45891	.15255
-7.10	.00000	.00000	.00	∞	.20720
-6.45	.00000	.00000	.08	.52512	.26550
-5.80	.00000	.00000	.16	.41855	.30224
-5.15	.00001	.00001	.32	.31957	.36129
-4.50	.00003	.00002	.64	.22838	.44751
-3.85	.00011	.00006	.96	.17803	.51239
-3.22	.00039	.00022	1.28	.14353	.56398
-2.90	.00073	.00041	1.60	.15279	.61139
-2.58	.00138	.00073	1.92	.14030	.65329
-2.25	.00262	.00128	2.25	.11609	.69931
-1.92	.00501	.00259	2.58	.06760	.72873
-1.60	.00968	.00492	2.90	.05650	.74857
-1.28	.01369	.00946	3.22	.04733	.76518
- .96	.03832	.01860	3.85	.03327	.79137
- .64	.08078	.03766	4.50	.02339	.80910
- .32	.18825	.08071	5.15	.01644	.82184
- .16	.32020	.12139			

B- X.

Parameter Values;

$\rho = 0.5, \quad V_1 = 0., \quad V_2 = 0.$

Z	f(z)	F(z)	Z	f(z)	F(z)
-5.250	.00000	.00000	.075	.70335	.49978
-4.650	.00001	.00001	.150	.53396	.54327
-4.050	.00005	.00003	.225	.44158	.58262
-3.450	.00018	.00015	.300	.37526	.61370
-2.850	.00055	.00031	.375	.32720	.63959
-2.550	.00114	.00083	.450	.28931	.66271
-2.250	.00205	.00105	.525	.25838	.68325
-2.100	.00297	.00173	.600	.23250	.70166
-1.950	.00408	.00210	.675	.21042	.71837
-1.800	.00528	.00259	.750	.19135	.73347
-1.650	.00827	.00332	.900	.15997	.76391
-1.500	.01155	.00417	1.050	.13526	.78235
-1.350	.01635	.00569	1.200	.11531	.80114
-1.200	.02328	.00764	1.350	.09895	.81683
-1.050	.03335	.01049	1.500	.08535	.83021
- .900	.04818	.01401	1.650	.07469	.84211
- .750	.07039	.02924	1.800	.06426	.85208
- .675	.08555	.03217	1.950	.05499	.86203
- .600	.10446	.04554	2.100	.04899	.86983
- .525	.12831	.05816	2.250	.04134	.87660
- .450	.15878	.07253	2.550	.03432	.88372
- .375	.19846	.09039	2.850	.02477	.89684
- .300	.25154	.11288	3.450	.01793	.93846
- .225	.32713	.14183	4.050	.01108	.95717
- .150	.43717	.17004	4.650	.00692	.96259
- .075	.63642	.18215	5.250	.00490	.96653
.000	∞	.23574			

B-XI.

Parameter Values;

$\rho = 0.5, \quad V_1 = 0.2, \quad V_2 = 0.45$

Z	f(z)	F(z)	Z	f(z)	F(z)
-5.700	.00000	.00000	.075	.73360	.39832
-5.100	.00000	.00000	.150	.51411	.43256
-4.500	.00001	.00001	.225	.42254	.50764
-3.900	.00002	.00002	.300	.36244	.53708
-3.300	.00026	.00004	.375	.31810	.56260
-3.000	.00053	.00026	.450	.28329	.57515
-2.700	.00099	.00039	.525	.25489	.60533
-2.400	.00174	.00080	.600	.23108	.62357
-2.250	.00250	.00122	.675	.21074	.64013
-2.100	.00344	.00156	.750	.18753	.65506
-1.950	.00475	.00218	.900	.16448	.67626
-1.800	.00724	.00308	1.050	.14189	.70031
-1.650	.00909	.00403	1.200	.12311	.72019
-1.500	.01264	.00593	1.350	.10725	.73646
-1.350	.01772	.00836	1.500	.09342	.75151
-1.200	.02485	.01142	1.650	.08207	.75467
-1.050	.03498	.01587	1.800	.07234	.77626
- .900	.04984	.02284	1.950	.06379	.78648
- .750	.06899	.03139	2.100	.05670	.80458
- .675	.08562	.03321	2.250	.05036	.80884
- .600	.10383	.03679	2.400	.04289	.81583
- .525	.12557	.04543	2.700	.03648	.83106
- .450	.15547	.05601	3.000	.02917	.84009
- .375	.19293	.06809	3.300	.02184	.84665
- .300	.24295	.085[illegible]2	3.900	.01656	.85117
- .225	.31302	.10627	4.500	.01085	.86939
- .150	.42092	.13369	5.100	.00717	.87279
- .075	.66379	.17344	5.700	.00546	.87859
.000	∞	.26058			

B-XII.

Parameter Values;

$\rho = 0.5, \quad V_1 = 0.7, \quad V_2 = 0.7$

Z	f(z)	F(z)	Z	f(z)	F(z)
-6.000	.00000	.00000	.075	.61742	.29582
-5.400	.00000	.00000	.150	.44530	.33567
-4.800	.00000	.00000	.225	.37479	.35643
-4.200	.00002	.00001	.300	.32876	.38280
-3.600	.00008	.00001	.375	.29478	.40161
-3.300	.00017	.00007	.450	.26801	.43729
-3.000	.00033	.00013	.525	.24603	.45657
-2.700	.00064	.00017	.600	.22746	.47369
-2.400	.00115	.00024	.675	.21143	.49007
-2.250	.00168	.00343	.750	.19737	.50611
-2.100	.00235	.00577	.825	.18487	.52061
-1.950	.00328	.00610	.900	.17366	.53389
-1.800	.00460	.00677	.975	.16354	.54654
-1.650	.00646	.00762	1.050	.15062	.55832
-1.500	.00811	.00879	1.200	.13884	.57619
-1.350	.01291	.01043	1.350	.12478	.60040
-1.200	.01838	.01279	1.500	.11265	.61760
-1.050	.02633	.01594	1.650	.10198	.63370
- .900	.03806	.02312	1.800	.09250	.64829
- .750	.05332	.04006	1.950	.08412	.66154
- .675	.06702	.05457	2.100	.06662	.67112
- .600	.08150	.06015	2.250	.07005	.68459
- .525	.10034	.06697	2.400	.06382	.69463
- .450	.12380	.07537	2.550	.05833	.70379
- .375	.15471	.08582	2.700	.04877	.71183
- .300	.19615	.09895	3.000	.04552	.72146
- .225	.25438	.11586	3.300	.03823	.72998
- .150	.34411	.13829	3.600	.03012	.73513
- .075	.54470	.17162	4.200	.02440	.75149
.000	∞	.21884	4.800	.01734	.76401
			5.400	.01234	.77290
			6.000	.01136	.78003

B-XIII.

Parameter Values;

$\rho = 0.5, \quad V_1 = 0.85, \quad V_2 = 0.85$

Z	f(z)	F(z)	Z	f(z)	F(z)
-7.200	.00000	.00000	- .150	.28920	.09971
-6.600	.00000	.00000	- .075	.41698	.15167
-6.000	.00000	.00000	.000	∞	.17293
-5.400	.00000	.00000	.075	.48290	.22214
-4.800	.00000	.00000	.150	.38721	.25477
-4.200	.00002	.00002	.300	.29921	.30625
-3.600	.00007	.00003	.600	.21902	.38216
-3.000	.00028	.00014	.900	.17488	.44297
-2.700	.00054	.00026	1.200	.14441	.49086
-2.400	.00104	.00052	1.500	.15747	.53617
-2.100	.00203	.00099	1.800	.14811	.58198
-1.800	.00397	.00198	2.100	.12553	.62314
-1.500	.00785	.00362	2.400	.07488	.65319
-1.200	.01552	.00714	2.700	.06410	.67404
- .900	.03259	.01434	3.000	.05499	.69191
- .600	.07038	.02980	3.600	.04056	.72056
- .300	.16799	.06556	4.200	.02992	.74171
			4.800	.02207	.75745